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LINEAR ALGEBRA AND ITS APPLICATIONS

A max version of the Perron–Frobenius theorem

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Abstract

If A is an $n \times n$ nonnegative, irreducible matrix, then there exists $\mu(A) > 0$, and a positive vector x such that $\max_j a_{ij}x_j = \mu(A)x_i, i = 1, 2, \ldots, n$. Furthermore, $\mu(A)$ is the maximum geometric mean of a circuit in the weighted directed graph corresponding to A. This theorem, which we refer to as the max version of the Perron–Frobenius Theorem, is well-known in the context of matrices over the max algebra and also in the context of matrix scalings. In the present work, which is partly expository, we bring out the intimate connection between this result and the Perron–Frobenius theory. We present several proofs of the result, some of which use the Perron–Frobenius Theorem. Structure of max eigenvalues and max eigenvectors is described. Possible ways to unify the Perron–Frobenius Theorem and its max version are indicated. Some inequalities for $\mu(A)$ are proved. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

There has been a great deal of interest in recent years in the algebraic system called "max algebra". This system allows one to express in a linear fashion, phenomena that are nonlinear in the conventional algebra. It has applications in many diverse areas such as parallel computation, transportation networks and scheduling. We refer to [1–4] for a description of such systems and their applications.

Although there are several abstract examples of a max algebra, the term is generally used to denote the set of reals, together with $-\infty$, equipped with the

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binary operations of maximization and addition. We prefer to deal with the set of nonnegative numbers, equipped with the binary operations of maximization and multiplication. This system is clearly isomorphic to the former one as the exponential map provides an isomorphism from the former system to the latter system.

Our interest will be in describing the analogue of the Perron–Frobenius theory for this new system, referred to as the *max version* of the theory. The theme of the paper is that the max version of the Perron–Frobenius theory complements the classical Perron–Frobenius theory and should be considered as an integral part of the classical theory. This paper contains a survey as well as some new results. Specifically, there is some novelty in the presentation of the proofs of Theorem 2 and the connection with the Frobenius–Victory Theorem pointed out in Section 2. Also, all the results in Sections 5,6 are new.

We consider the set of nonnegative numbers equipped with two binary operations, defined as follows. If $a \geqslant 0, b \geqslant 0$, then their sum, denoted $a \oplus b$, is defined as $\max(a, b)$ whereas their product is the usual product, ab. Addition and multiplication of vectors and matrices are defined in a natural way. If A, B are matrices compatible for matrix multiplication, then we denote their product by $A \otimes B$, when $\oplus$ is used as a sum, to distinguish it from AB. For example, if

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}.$$

Then

$$A \oplus B = \begin{bmatrix} 3 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}, \text{ and } \quad A \otimes B = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 1 & 4 \\ 3 & 2 & 4 \end{bmatrix}.$$

It can be easily proved that the product $\otimes$ is associative and that it distributes over the sum $\oplus$.

The paper is organized as follows. In Section 2 we present several proofs of the max version of the Perron–Frobenius Theorem, some of which are new and some, although known, are presented for completeness. In the next two sections we describe the structure of the max eigenvalues and eigenvectors of a nonnegative matrix. In Section 5 we show possible ways to unify the Perron–Frobenius Theorem and its max version. In Section 6 some inequalities are proved, continuing a program initiated in [4,5].

2. Proofs of the max version

If A is an $n \times n$ nonnegative matrix, then $\mathscr{G}(A)$ will denote the weighted directed graph associated with A; thus $\mathscr{G}(A)$ has vertices $1, 2, \ldots, n$, and there is an edge from i to j with weight a_{ij} if and only if $a_{ij} > 0$. By a circuit we always mean a simple circuit. The class of circuits includes *loops*, i.e., circuits of length 1. In contrast, our paths may include a vertex and/or an edge more than once. By the *product of a path* we mean the product of the weights of the edges in the path.

If $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ is a circuit in $\mathscr{G}(A)$, then $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ is the corresponding *circuit product* and its kth root is a *circuit geometric mean*. The maximum circuit geometric mean in $\mathscr{G}(A)$ will be denoted by $\mu(A)$. A circuit is called *critical* if the corresponding circuit product equals $\mu(A)$. An edge (or a vertex) of $\mathscr{G}(A)$ is critical if it belongs to a critical circuit. The subgraph of $\mathscr{G}(A)$ spanned by the critical vertices is known as the *critical graph* of A.

An $n \times n$ nonnegative matrix A is called a *circuit matrix* (see, for example, [3,6]) if each entry of the matrix is either 0 or 1 and if $\mathscr{G}(A)$ is a circuit.

We assume familiarity with basic aspects of nonnegative matrices and refer to [7] for concepts such as the Frobenius Normal Form, basic and final classes etc., which are not explicitly defined in this paper.

The following will be needed in the sequel.

Lemma 1. *Let A be an $n \times n$ nonnegative, irreducible matrix and suppose $x \in \mathbb{R}^n, x \geqslant 0, x \neq 0, \gamma > 0$ such that $A \otimes x = \gamma x$. Then $x > 0$ and $\gamma = \mu(A)$.*

Proof. The proof of the fact that $x > 0$ is similar to the well-known assertion that *a nonnegative eigenvector of an irreducible, nonnegative matrix must be positive* (see, for example, [8], p. 7) and is omitted. We then have

$$\max_j a_{ij} x_j = \gamma x_i, \quad i = 1, 2, \ldots, n. \tag{1}$$

If $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ is a circuit in $\mathscr{G}(A)$, then by (1),

$$a_{i_s i_{s+1}} x_{i_{s+1}} \leqslant \gamma x_{i_s}, \quad s = 1, 2, \ldots, k,$$

where $k + 1$ is taken to be 1. It follows that the corresponding circuit geometric mean is at most γ and thus we have shown that $\gamma \geqslant \mu(A)$.

Now consider the subgraph of $\mathscr{G}(A)$ which consists of those (i, j) for which $a_{ij} x_j = \gamma x_i$. In this graph each vertex has out degree at least one and hence it contains a circuit. That circuit must have circuit geometric mean equal to γ and therefore $\gamma = \mu(A)$. □

We refer to the next result as the max version of the Perron–Frobenius Theorem.

Theorem 2. *Let A be an $n \times n$ nonnegative, irreducible matrix. Then there exists a positive vector x such that $A \otimes x = \mu(A)x$.*

We now present several proofs of this theorem. The main purpose behind presenting these proofs is that they may provide a better understanding of the result and suggest some extensions. Proof 1 is known (see, for example, [1], p. 185–187) and is sketched here for completeness. Proof 2, using Brouwer's Fixed Point Theorem, imitates the corresponding proof of the Perron–Frobenius Theorem. Proofs 3 and 4 derive the max version from the Perron–Frobenius Theorem itself. Finally, Proof 5, based on the duality theorem, is hinted in [1], p. 205 but is worked out here rigorously. We denote the Perron eigenvalue of the nonnegative matrix A by $\rho(A)$.

Proof 1. We assume, without loss of generality, that $\mu(A) = 1$, for otherwise we may consider the matrix $1/\mu(A)A$. Let $\Gamma(A) = [\gamma_{ij}]$ be the $n \times n$ matrix where γ_{ij} is the maximum product of a path from i to j in $\mathscr{G}(A)$. (Since $\mu(A) = 1, \gamma_{ij} < \infty$.) For any $i, k \in \{1, 2, \ldots, n\}$ we have

$$(A \otimes \Gamma)_{ik} = \max_j a_{ij}\gamma_{jk} \leqslant \gamma_{ik} \tag{2}$$

and therefore $A \otimes \Gamma \leqslant \Gamma$, where the inequality is to be interpreted componentwise. Now let $k \in \{1, 2, \ldots, n\}$ be a critical vertex and let Γ_k denote the kth column of $\Gamma(A)$. Clearly, $\gamma_{kk} = 1$ and hence $\Gamma_k \neq 0$. Observe that $\max_j a_{ij}\gamma_{jk}$ represents the maximum product of a path from i to k where the maximum is taken over paths of length at least 2. Since k is critical, any path from i to k can be augmented by a circuit from k to itself with circuit product 1 and thus equality must hold in the inequality in (2). Thus $A \otimes \Gamma_k = \Gamma_k$. By Lemma 1, $\Gamma_k > 0$ and the proof is complete. □

Proof 2. Let $\mathscr{P} = \{y \in \mathbb{R}^n\colon\ y \geqslant 0, \sum_{i=1}^n y_i = 1\}$. Define the map $f : \mathscr{P} \to \mathscr{P}$ as

$$f(y) = \left\{ \sum_{i=1}^n (A \otimes y)_i \right\}^{-1} A \otimes y,$$

if $y \in \mathscr{P}$. Since A is irreducible, $f(y)$ is well-defined for any $y \in \mathscr{P}$. Also, f is continuous and hence by Brouwer's Fixed Point Theorem, there exists $x \in \mathscr{P}$ such that $f(x) = x$. Therefore

$$A \otimes x = \left\{ \sum_{i=1}^n (A \otimes x)_i \right\} x.$$

It follows from Lemma 1 that

$$\mu(A) = \sum_{i=1}^n (A \otimes x)_i,$$

completing the proof. □

Proof 3. Let $\hat{A}$ be an $n \times n$ matrix obtained by choosing one positive entry in each row of A and with maximum Perron eigenvalue. Then it can be seen that

$\rho(\hat{A})$, the Perron eigenvalue of $\hat{A}$ is $\mu(A)$. Write $\hat{A}$ in the Frobenius Normal Form

$$\begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}.$$

If class i is basic then B_{ii} must be a circuit matrix and so class i is final. If class i is not basic, then it must be a 1×1 zero matrix, in which case it is not final. Thus in the Frobenius Normal Form, the basic classes are the same as the final classes and hence (see [7], p. 40) $\hat{A}$ admits a positive eigenvector x corresponding to $\mu(A)$. Thus $\hat{A}x = \mu(A)x$. It remains to show that $\max_j a_{ij}x_j = \mu(A)x_i, i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$ let $j_i, k_i \in \{1, 2, \ldots, n\}$ be such that the (i, j_i) entry in $\hat{A}$ is positive and

$$\max_j a_{ij}x_j = a_{ik_i}x_{k_i}, \quad i = 1, 2, \ldots, n.$$

Let $\tilde{A}$ be the matrix with the (i, k_i) entry equal to $a_{ik_i}, i = 1, 2, \ldots, n$, and the remaining entries equal to zero. Then

$$\tilde{A}x \geqslant \hat{A}x = \mu(A)x. \tag{3}$$

Premultiplying (3) by a Perron eigenvector of $\tilde{A}$ we see that $\rho(\tilde{A}) \geqslant \mu(A)$. However, by construction of $\hat{A}, \mu(A) = \rho(\hat{A}) \geqslant \rho(\tilde{A})$. Since x is positive, it follows that equality must hold in (3) and the proof is complete. □

Proof 4. This proof also uses the Perron–Frobenius Theorem. For any positive integer k, let $A^{(k)}$ denote the matrix $[a_{ij}^k]$. A similar notation applies to vectors. Let $y_{(k)}$ denote a Perron eigenvector of $A^{(k)}$ and let $z_{(k)}$ denote the vector $y_{(k)}^{(k)}$ normalized to a probability vector (i.e. a nonnegative vector with components adding up to 1.) The sequence $z_{(1)}, z_{(2)}, \ldots$ belongs to the compact set of probability vectors and hence admits a convergent subsequence. For convenience we denote the subsequence again by $z_{(k)}, k = 1, 2, \ldots$ Thus we assume that $\lim_{k \to \infty} z_{(k)}$ exists and we denote it by x. We have

$$\sum_{j=1}^{n} a_{ij}^k z_{(k)j}^k = \rho(A^{(k)}) z_{(k)i}^k, \quad i = 1, 2, \ldots, n.$$

Therefore

$$\left\{ \sum_{j=1}^{n} a_{ij}^k z_{(k)j}^k \right\}^{1/k} = (\rho(A^{(k)}))^{1/k} z_{(k)i}, \quad i = 1, 2, \ldots, n.$$

It is known (see, for example, [9]) that $(\rho(A^{(k)}))^{1/k} \to \mu(A)$ as $k \to \infty$. Thus letting k go to infinity in the equation above, we get $\max_j a_{ij}x_j = \mu(A)x_i, i =$

$1, 2, \dots, n$. It follows from Lemma 1 that x must be positive and the proof is complete. □

Proof 5. This proof uses the duality theorem of linear programming. Let $b_{ij} = \log a_{ij}$ if $a_{ij} > 0$ and $b_{ij} = -\Delta$ if $a_{ij} = 0$, where Δ is sufficiently large so that the critical graph of A is the same as that of $\left[e^{b_{ij}}\right]$. Consider the linear programming problem:

$$\text{minimize } \tau \text{ subject to } b_{ij} + y_j - y_i \leqslant \tau, \quad i, j = 1, 2, \dots, n.$$

The dual problem is

$$\max \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} w_{ij}$$

$$\text{s.t. } w_{ij} \geqslant 0, \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} = 1, \quad \sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ji}, \quad i = 1, 2, \dots, n.$$

Since both these problems are clearly feasible, by the duality theorem, they both have optimal solutions. Let $\tau_0, y_1, \dots, y_n$ be optimal for the primal problem. We recall the known result (see, for example, [10]) that the set of feasible matrices $W = [w_{ij}]$ for the dual problem is precisely the set of nonnegative, nontrivial linear combinations of circuit matrices. (Such matrices are called sum-symmetric matrices and we will encounter them again in Section 6.) Thus the optimal solution for the dual (and hence the primal) problem is precisely the maximum circuit arithmetic mean in B. It follows that $\mu(A) = e^{\tau_0}$. Let $x_i = e^{y_i}, i = 1, 2, \dots, n$. Then we have

$$a_{ij} x_j \leqslant \mu(A) x_i, \quad i, j = 1, 2, \dots, n. \tag{4}$$

Let $S \subset \{1, 2, \dots, n\}$ be a nonempty set such that

$$\max_j a_{ij} x_j = \mu(A) x_i \tag{5}$$

if and only if $i \in S$. We assume, without loss of generality, that S is the maximal subset (i.e., has maximum cardinality) with this property. Let, if possible, $S \neq \{1, 2, \dots, n\}$. Then we may increase each $x_i, i \in S$ by a suitable constant so that the resulting $x_1, \dots, x_n$ still satisfy (4), while (5) holds for some $i \notin S$, which is a contradiction. Therefore $S = \{1, 2, \dots, n\}$ and hence $A \otimes x = \mu(A) x$. □

Theorem 2 can be thought of as a result on matrix scalings. Thus if A is a nonnegative, irreducible matrix, then Theorem 2 is equivalent to the assertion that there exists a diagonal matrix D with positive diagonal entries such that in $D^{-1}AD$, the maximum entry in each row is the same. (Indeed, using the notation of Theorem 2, the ith diagonal element of D is $x_i, i = 1, 2, \dots, n$.) In this context we refer to [10,11] for some related results. Several characterizations of $\mu(A)$ have also been given in [12].

3. Max eigenvalues of a reducible matrix

Let A be an $n \times n$ nonnegative matrix. We say that λ is a *max eigenvalue* of A if there exists a nonzero, nonnegative vector x such that $A \otimes x = \lambda x$. We refer to x as a corresponding *max eigenvector*. As seen in the previous section, if A is irreducible, then it admits $\mu(A)$ as the unique max eigenvalue. The situation is more interesting when A is reducible. Consider the following matrices.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}. \tag{6}$$

Clearly, A has two max eigenvalues, 4, 5, with $(1,0)^{\mathrm{T}}, (0,1)^{\mathrm{T}}$ as the corresponding eigenvectors. It can be verified that B has 5 as the only max eigenvalue, whereas C has two max eigenvalues, 4, 5.

The general result describing the max eigenvalues of an arbitrary nonnegative matrix, obtained by Gaubert [13], and independently by Wende et al. [14], is given next.

Theorem 3. *Let A be an $n \times n$ nonnegative matrix in Frobenius Normal Form,*

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}.$$

Then λ is a max eigenvalue of A if and only if there exists $i \in \{1, 2, \ldots, m\}$ such that $\lambda = \mu(A_{ii})$ and furthermore, class j does not have access to class i whenever $\mu(A_{jj}) > \mu(A_{ii})$.

It is easy to figure out the max eigenvalues of the matrices in (6) using Theorem 3. Thus B does not have 4 as a max eigenvalue since the second class, which has a higher circuit geometric mean, has access to the first class.

We now wish to point out that Theorem 3 can be regarded as the max version of a result in classical Perron–Frobenius theory, which goes back to Frobenius, and which has been termed as the Frobenius–Victory Theorem by Schneider [15].

Let A be a nonnegative $n \times n$ matrix. Which eigenvalues of A admit a nonnegative eigenvector? Clearly, if A is irreducible, then the Perron eigenvalue $\rho(A)$ is the only eigenvalue with this property. In the general case, we have the following, see [15].

Theorem 4 (Frobenius–Victory Theorem). *Let A be an $n \times n$ nonnegative matrix in Frobenius Normal Form,*

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}.$$

Then λ is an eigenvalue of A with a corresponding nonnegative eigenvector if and only if there exists $i \in \{1, 2, \ldots, m\}$ such that $\lambda = \rho(A_{ii})$ and furthermore, class j does not have access to class i whenever $\rho(A_{jj}) > \rho(A_{ii})$.

Some remarks are in order now. If we work exclusively over the semiring of nonnegative numbers, then an eigenvalue of a (nonnegative) matrix A should be rightfully defined as a number λ such that there exists a nonzero, nonnegative vector x with $Ax = \lambda x$. Taking this viewpoint we realize that Theorem 4 is merely describing the set of "eigenvalues" of a nonnegative matrix. The second remark is that Theorem 3 should be regarded as a max version of Theorem 4. In fact if we apply Theorem 4 to the matrix $A^{(k)} = [a_{ij}^k]$ and use a limiting argument as in Proof 4 of Theorem 2, then we deduce the "if" part of Theorem 3.

4. Max eigenvectors

Let A be a nonnegative, irreducible $n \times n$ matrix. Then it is well-known that the eigenvector corresponding to $\rho(A)$ is unique up to a scalar multiple. The situation concerning the max eigenvectors is more interesting.

We assume, without loss of generality, that $\mu(A) = 1$. Recall the definitions of critical circuit and critical graph in Section 2. We also use the matrix Γ introduced in Proof 1 of Theorem 2. We denote the critical graph of A by $\mathscr{H}(A)$. The structure of the max eigenvectors is described in the next result.

Theorem 5. *Let A be a nonnegative, irreducible $n \times n$ matrix with $\mu(A) = 1$. Suppose the critical graph $\mathscr{H}(A)$ has k strongly connected components and let $V_1, \ldots, V_k$ be the corresponding sets of vertices. Then the following assertions hold.*

1. *Any column of Γ corresponding to a vertex of $\mathscr{H}(A)$ is a max eigenvector of A.*
2. *Any two columns of Γ corresponding to vertices in the same V_i are scalar multiples of each other.*
3. *Let $i_j \in V_j$, and let Γ_{i_j} be the corresponding column of $\Gamma, j = 1, 2, \ldots, k$. Then $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ are "linearly independent" in the sense that none of them is a linear combination (using the max sum) of others. Furthermore, any eigenvector of A is a linear combination (again using the max sum) of $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$.*

Example 6. Consider the matrix

$$A = \begin{bmatrix} 0.5 & 1 & 0.3 & 0.4 & 0.2 \\ 1 & 0.2 & 0 & 0 & 0.3 \\ 1 & 0 & 0 & 1 & 0 \\ 0.2 & 1 & 1 & 0.3 & 0.2 \\ 0 & 0.4 & 1 & 0 & 1 \end{bmatrix}.$$

Then $\mu(A) = 1$ and the critical graph $\mathscr{H}(A)$ has 3 strongly connected components. It can be verified that

$$\Gamma = \begin{bmatrix} 1 & 1 & 0.4 & 0.4 & 0.3 \\ 1 & 1 & 0.4 & 0.4 & 0.3 \\ 1 & 1 & 1 & 1 & 0.3 \\ 1 & 1 & 1 & 1 & 0.3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We conclude that $(1, 1, 1, 1, 1)^{\mathrm{T}}$, $(0.4, 0.4, 1, 1, 1)^{\mathrm{T}}$ and $(0.3, 0.3, 0.3, 0.3, 1)^{\mathrm{T}}$ constitute a set of "linearly independent" max eigenvectors of A and any other max eigenvector is a linear combination (using the max sum) of these vectors.

For a proof of Theorem 5 and for a similar result for reducible matrices we refer to [1,14]. We conclude this section by stating a simple consequence of Theorem 5.

Corollary 7. *Let A be a nonnegative, irreducible $n \times n$ matrix. Then the max eigenvector of A is unique, up to a scalar multiple, if and only if the critical graph $\mathscr{H}(A)$ is strongly connected.*

5. Unifying the Perron–Frobenius Theorem and its max version

If x is a vector of order n and if $0 < p \leqslant \infty$, then let $||x||_p$ denote the ℓ_p-norm of x, given by $\{\sum_{i=1}^{p} |x_i|^p\}^{1/p}$. Then the following result clearly unifies the Perron–Frobenius Theorem and its max version. The proof can be given using Brouwer's Fixed Point Theorem as before.

Theorem 8. *Let A be a nonnegative, irreducible, $n \times n$ matrix and let $0 < p \leqslant \infty$. Then there exists $\lambda > 0$ and a nonnegative vector x such that*

$$||(a_{i1}x_1, \ldots, a_{in}x_n)||_p = \lambda x_i, \quad i = 1, \ldots, n.$$

Note that if $0 < p < \infty$, then Theorem 7 may be derived by applying the Perron–Frobenius Theorem to the matrix $[a_{ij}^p]$.

We now indicate another possible way to unify the two results. Let k be a fixed positive integer, $1 \leqslant k \leqslant n$. If $x, y \in \mathbb{R}^n$, then let us define $x^{\mathrm{T}} \otimes_k y$ to be the sum of the k largest components in $\{x_1 y_1, \ldots, x_n y_n\}$. If A is an $n \times n$ matrix and if $x \in \mathbb{R}^n$, then $A \otimes_k x$ is then defined in the obvious way, i.e., the ith component of $A \otimes_k x$ is given by $(a_{i1}, \ldots, a_{in}) \otimes_k x$. Observe that when $k = n$, $x^{\mathrm{T}} \otimes_k y$ is simply $\sum_{i=1}^{n} x_i y_i$, whereas if $k = 1$, then $x^{\mathrm{T}} \otimes_1 y = x^{\mathrm{T}} \otimes y = \max_j x_j y_j$. Thus the following result may be thought of as a unification of the Perron–Frobenius Theorem and its max version. The proof of the result can be given, using Brouwer's Fixed Point Theorem, along the lines of Proof 2 of Theorem 2 and is omitted. We only make one remark which is useful in the proof. If A is a nonnegative, irreducible $n \times n$ matrix and if $x \in \mathbb{R}^n$ is a nonnegative, nonzero vector such that $A \otimes_k x$ is a constant multiple of x, then it follows that $x > 0$.

Theorem 9. *Let A be an $n \times n$ nonnegative, irreducible matrix and let $1 \leqslant k \leqslant n$. Then there exists a positive vector x and a constant γ such that $A \otimes_k x = \gamma x$.*

We now turn to the question of uniqueness of γ in Theorem 9. If A is an $n \times n$ nonnegative matrix, let $\mu_k(A)$ denote the maximum Perron eigenvalue of a matrix obtained by retaining at most k nonzero entries in each row.

Theorem 10. *Let A be an $n \times n$ nonnegative, irreducible matrix, let $1 \leqslant k \leqslant n$ and let $\gamma > 0, x > 0$ such that $A \otimes_k x = \gamma x$. Then $\gamma = \mu_k(A)$.*

Proof. Let $\hat{A}$ denote a matrix obtained by retaining at most k nonzero entries in each row of A. Then clearly $\hat{A}x \leqslant \gamma x$. Premultiplying this inequality by a left Perron eigenvector of $\hat{A}$ and keeping in mind that $x > 0$, we get $\rho(\hat{A}) \leqslant \gamma$. Also, since the sum of the largest k components in $\{a_{i1}x_1, \ldots, a_{in}x_n\}$ is $\gamma x_i, i = 1, 2, \ldots, n$, we can make a choice of $\hat{A}$ for which $\gamma = \rho(\hat{A})$. It follows that $\gamma = \mu_k(A)$. □

It is instructive to have a graph theoretic interpretation of $\mu_k(A)$. By the Perron eigenvalue of a directed graph we mean the Perron eigenvalue of its adjacency matrix. Observe that $\mu_k(A)$ is precisely the maximum Perron eigenvalue of a strongly connected subgraph of $\mathscr{G}(A)$ in which the outdegree of any vertex is at most k.

It appears that further development of a Perron–Frobenius type theory is difficult when $1 < k < n$. We point towards some problems that arise.

In the classical Perron–Frobenius theory, the Perron eigenvalue of A and A^{T} is the same, and this fact is crucially used in the theory. A similar result holds in the max version, i.e., the case $k = 1$. However, when $1 < k < n$, the property is no longer valid as indicated by the next example.

Example 11. Let

$$A = \begin{bmatrix} 0 & 6 & 3 & 4 & 0 \\ \frac{11}{2} & 3 & 0 & \frac{1}{2} & \frac{9}{2} \\ 7 & 0 & 3 & 0 & 0 \\ 7 & 3 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 & 0 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad y = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $A \otimes_2 x = 10x$ and $A^{\mathrm{T}} \otimes_2 y = 9y$. Thus $\mu_2(A) = 10$ whereas $\mu_2(A^{\mathrm{T}}) = 9$.

We may be able to take care of the problem indicated in Example 11 by imposing pattern conditions on the matrix. One example of such a result is the following.

Theorem 12. *Let A be an $n \times n$ nonnegative, irreducible, tridiagonal matrix. Then $\mu_k(A) = \mu_k(A^{\mathrm{T}}), k = 1, 2, \ldots, n$.*

Proof. If $k \geqslant 3$ then $\mu_k(A)$ and $\mu_k(A^{\mathrm{T}})$ are both equal to the Perron eigenvalue of A, whereas if $k = 1$, then $\mu_1(A) = \mu_1(A^{\mathrm{T}}) = \mu(A)$. Thus we only consider the case $k = 2$. Since A is tridiagonal, a strongly connected subgraph of $\mathscr{G}(A)$ must be a subgraph induced by vertices $i, i+1, \ldots, j$ for some $i \leqslant j$. Thus a strongly connected subgraph in which each vertex has outdegree at most two, consists of a subgraph induced by vertices $i, i+1$ for some i or of a single loop. It follows in view of the remark following Theorem 10 that

$$\mu_2(A) = \max\left\{a_{ii}, i = 1, 2, \ldots, n;\ \rho\begin{bmatrix} a_{ii} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{bmatrix}, \ i = 1, 2, \ldots, n-1\right\}.$$

As a consequence of this formula for $\mu_2(A)$, we conclude that $\mu_2(A) = \mu_2(A^{\mathrm{T}})$. □

In the remainder of this section we keep $k \in \{1, 2, \ldots, n\}$ fixed. We introduce terminology which is effective only in the present section. If A is a nonnegative, irreducible $n \times n$ matrix and if $A \otimes_k x = \mu_k(A)x$, where x is a nonzero, nonnegative vector, then we refer to x as a *k-eigenvector* of A. The set of k-eigenvectors of A is closed under the usual addition if $k = n$ and under $\oplus$ if $k = 1$. However, it is not closed under either of these additions if $1 < k < n$ and thus we cannot think of a concept such as a basis for the eigenspace. We now give a condition under which the eigenvector is unique, up to a scalar multiple, thereby generalizing the result known for $k = 1$, n (see Corollary 7).

We call a subgraph of $\mathscr{G}(A)$ *k-maximal* if it has maximum Perron eigenvalue among all strongly connected subgraphs with each vertex having outdegree at most k. The union of all k-maximal subgraphs is called the *k-critical subgraph*. A vertex or an edge is said to be *k-critical* if it belongs to the k-critical subgraph.

Theorem 13. *Let A be a nonnegative, irreducible $n \times n$ matrix and suppose the k-critical subgraph is strongly connected. Then there is a unique k-eigenvector, up to a scalar multiple.*

Proof. Let x, y be eigenvectors, so that $A \otimes_k x = \mu_k(A)x, A \otimes_k y = \mu_k(A)y$. Since A is irreducible, x, y must be positive. There exists a matrix $\hat{A}$ obtained by retaining at most k positive entries in each row of A such that $\hat{A}x = \mu_k(A)x$. Clearly,

$$\hat{A}y \leqslant A \otimes_k y = \mu_k(A)y. \tag{7}$$

Premultiplying (7) by a Perron eigenvector of $\hat{A}$ we get $\mu_k(\hat{A}) \leqslant \mu_k(A)$. Since $\mu_k(\hat{A}) = \mu_k(A)$, we must have $\hat{A}y = \mu_k(A)y$. We assume, without loss of generality, that $\hat{A}$ is in the Frobenius Normal Form. Since $\hat{A}$ admits a positive Perron eigenvector, its basic classes and the final classes are the same (see [7], p. 40). Let

$$\hat{A} = \begin{bmatrix} B_{11} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & B_{mm} & 0 & \cdots & 0 \\ B_{m+1,1} & \cdots & B_{m+1,m} & B_{m+1,m+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pm} & B_{p,m+1} & \cdots & B_{pp} \end{bmatrix},$$

where classes $1, 2, \dots, m$ are the basic, final ones. Since $B_{ii}, i = 1, 2, \dots, m$ are irreducible, the corresponding subvectors in x, y must be proportional.

Thus if the edges $(i, j), (k, \ell)$ of $\mathscr{G}(A)$ belong to the same k-maximal subgraph, then (x_i, x_k) is proportional to (y_j, y_ℓ). Since the k-critical subgraph is strongly connected, it follows that the subvectors of x, y corresponding to the k-critical vertices in $\mathscr{G}(A)$ are proportional. The remaining parts of x, y are then completely determined (as in the proof of Theorem 3.10 in [7]) and they must also be proportional. □

6. Inequalities for $\mu(A)$

In this section we first continue a program initiated in [4] and obtain an inequality for $\mu(A)$ which is motivated by known inequalities for nonnegative matrices. The inequality is similar to Bŕegman inequality [16] concerning the permanent. Yet another Bŕegman type inequality was proved in [5].

An $n \times n$ matrix is called *sum-symmetric* if its ith row sum equals the ith column sum for $i = 1, 2, \dots, n$. We first prove a preliminary result.

Lemma 14. *Let Z be an $n \times n$ positive matrix. Then for any sum-symmetric matrix $U = [u_{ij}]$ with $\sum_{i,j=1}^n u_{ij} = 1$, it is true that*

$$\sum_{i,j=1}^n u_{ij} \log z_{ij} \leqslant \log \mu(Z).$$

Proof. If U is a sum-symmetric matrix with $\sum_{i,j=1}^n u_{ij} = 1$, then it is well-known (see [3], p. 126) that U can be expressed as a nonnegative linear combination of circuit matrices. Let $U = \sum_{k=1}^p \alpha_k C^{(k)}$ where $\alpha_k \geqslant 0$ and $C^{(k)} = [c_{ij}^{(k)}]$ is a circuit matrix, $k = 1, 2, \ldots, p$. Let ℓ_k be the length of the circuit corresponding to $C^{(k)}, k = 1, 2, \ldots, p$. Clearly,

$$\sum_{i,j=1}^n c_{ij}^{(k)} \log z_{ij} \leqslant \ell_k \log \mu(Z). \tag{8}$$

It is easily seen, using $\sum_{i,j=1}^n u_{ij} = 1$, that

$$\sum_{k=1}^p \alpha_k \ell_k = 1. \tag{9}$$

Now

$$\begin{aligned}\sum_{i,j=1}^n u_{ij} \log z_{ij} &= \sum_{i,j=1}^n \left(\sum_{k=1}^p \alpha_k c_{ij}^{(k)} \right) \log z_{ij} \\ &= \sum_{k=1}^p \alpha_k \sum_{i,j=1}^n c_{ij}^{(k)} \log z_{ij} \leqslant \log \mu(Z),\end{aligned}$$

where the inequality follows from (8) and (9). That completes the proof. □

As before, $\mathscr{K}(A)$ will denote the critical graph of A.

Theorem 15. *Let A be a nonnegative $n \times n$ matrix and suppose $\mu(A) > 0$. Let U be an $n \times n$ sum-symmetric matrix such that $\sum_{i,j=1}^n u_{ij} = 1$ and suppose that $u_{ij} = 0$ whenever $(i,j) \notin \mathscr{K}(A)$. Then for any nonnegative matrix Z,*

$$\mu(Z) \geqslant \mu(A) \prod_{i,j=1}^n \left(\frac{z_{ij}}{a_{ij}} \right)^{u_{ij}}. \tag{10}$$

Proof. We follow the proof technique used to prove Theorem 3.2 in [5]. We assume that A, Z are positive as the general case may then be obtained by a continuity argument.

Inequality (10) is clearly equivalent to

$$\log \mu(Z) \geqslant \log \mu(A) + \sum_{i,j=1}^n u_{ij} (\log z_{ij} - \log a_{ij}). \tag{11}$$

As in Proof 5 of Theorem 2, let $b_{ij} = \log a_{ij}$ for all i, j, and consider the linear programming problem:

$$\text{minimize } \tau \quad \text{subject to} \quad b_{ij} + y_j - y_i \leqslant \tau, \ \ i, j = 1, 2, \ldots, n.$$

The dual problem is

$$\max \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} w_{ij}$$

$$\text{s.t. } w_{ij} \geqslant 0, \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} = 1, \quad \sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ji}, \ \ i = 1, 2, \ldots, n.$$

Choose optimal $x_1, \ldots, x_n$ for the primal problem. Then

$$b_{ij} + x_j - x_i \leqslant \log \mu(A), \quad i, j = 1, 2, \ldots, n \tag{12}$$

and equality holds if (i, j) is on a critical circuit.

Now

$$\begin{aligned} \sum_{i,j=1}^{n} u_{ij}(\log z_{ij} - \log a_{ij}) &= \sum_{i,j=1}^{n} u_{ij}(\log z_{ij} - \log \mu(A) - x_i + x_j) \\ &= \sum_{i,j=1}^{n} u_{ij} \log z_{ij} - \log \mu(A) \sum_{i,j=1}^{n} u_{ij} \leqslant \log \mu(Z) \\ &\quad - \log \mu(A), \end{aligned}$$

where the first equality comes from the fact that equality holds in (12) if $(i, j) \in \mathscr{H}(A)$ while $u_{ij} = 0$ if $(i, j) \notin H(A)$. The second equality is then obvious. Finally, the last inequality follows from Lemma 14 and the fact that $\sum_{i,j=1}^{n} u_{ij} = 1$. That completes the proof of (11) and hence the theorem. □

Our next result is motivated by the fact (see Proof 4 of Theorem 2) that for an $n \times n$ nonnegative matrix A, $(\rho(A^{(k)}))^{1/k} \to \mu(A)$ as $k \to \infty$.

Theorem 16. *Let A be an $n \times n$ nonnegative matrix. Then*

$$\lim_{k \to \infty} (\mu(A^k))^{1/k} = \rho(A). \tag{13}$$

Proof. We remark that the power A^k is the usual kth power of a matrix, rather than the power obtained by using the *max* addition.

If B is a nonnegative $n \times n$ matrix, then (see [12]), for any integer $k \geqslant 0$, $\mu(B) \leqslant \rho(B)$.

Therefore

$$(\rho(A))^k = \rho(A^k) \geqslant \mu(A^k)$$

and hence

$$(\mu(A^k))^{1/k} \leqslant \rho(A). \tag{14}$$

If A is an $n \times n$ nonnegative matrix, then it is well-known (see, for example, [12]) that

$$\rho(A) \leqslant n\mu(A)$$

and therefore

$$\rho(A^k) \leqslant n\mu(A^k).$$

It follows that

$$\left(\frac{1}{n}\right)^{1/k} \rho(A) \leqslant (\mu(A^k))^{1/k}. \tag{15}$$

The result follows by letting $k \to \infty$ in (14) and (15). □

It is also known (see [17]) that for an $n \times n$ nonnegative matrix A, $(\rho(A^{(p)}))^{1/p} \leqslant (\rho(A^{(q)}))^{1/q}$ for $0 < p < q$. Thus it is tempting to conjecture that

$$(\mu(A^p))^{1/p} \leqslant (\mu(A^q))^{1/q} \tag{16}$$

for positive integers $p < q$. This is false however, as indicated by the next example.

Example 17. Let

$$A = \begin{bmatrix} 1 & 6 \\ 3 & 0 \end{bmatrix}.$$

Then it can be verified that

$$(\mu(A^2))^{1/2} \approx 4.3589 \quad \text{while} \quad (\mu(A^3))^{1/3} \approx 4.3198.$$

In our concluding result we show that (16) is true if $p = 1$.

Theorem 18. *Let A be an $n \times n$ nonnegative matrix. Then for any positive integer q,*

$$\mu(A) \leqslant (\mu(A^q))^{1/q}. \tag{17}$$

Proof. Suppose

$$\mu(A) = (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_\ell i_1})^{1/\ell}.$$

Consider the closed path of length $q\ell$ obtained by going around the circuit

$$i_1 \to i_2 \to \cdots \to i_\ell \to i_1,$$

q times. The corresponding path product is

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_\ell i_1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_\ell i_1} \cdots a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_\ell i_1}.$$

Rewrite this as $\alpha_1\alpha_2\cdots\alpha_\ell$, where α_1 is the product of the first q terms, α_2 is the product of the next q terms and so on. Then it can be seen that

$$(\alpha_1\alpha_2\cdots\alpha_\ell)^{1/\ell} \leqslant \mu(A^q). \tag{18}$$

Since $\alpha_1\alpha_2\cdots\alpha_\ell = (\mu(A))^{q\ell}$, the result follows from (18). □

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